International Journal of Theoretical Physics, Vol. 31, No. 2, 1992

Binary Logic is Rich Enough

Roman R. Zapatrin[1]

Received March 14, 1991

Given a finite ortholattice L, the *-semigroup is explicitly built whose annihilator ortholattice is isomorphic to L. Thus, it is shown that any finite quantum logic is the additive part of a binary logic. Some areas of possible applications are outlined.

INTRODUCTION

Binary logic has been introduced as the underlying structure for quantum logics. Within this approach a physical object is associated with a semigroup. Each element of this semigroup, called generating, is understood as a conceivable elementary coercion performed upon the object (Zapatrin, 1989). A collection of elementary coercions is outlined, called the absurd subset, whose elements are understood as unperformable (under given obstacles). The mathematical structure describing binary logics is the linear logic (Girard, 1987) which operates with certain subsets of the semigroup, called facts, and possesses logical operations of two kinds: multiplicative and additive ones. While the multiplicative logic is the special feature, the additive part of binary logic is a usual ortholattice, which can be considered as a quantum logic in its conventional sense. The traditional quantum logic has a drawback which is eliminated within the binary logic approach. When an object is identified with the collection of all its observed properties (as is done in quantum logic) the problem of identification of the same object in different obstacles (when different sets of observation means are in disposal) arises. Using the binary logic approach, the semigroup itself stays unchanged, while the variations of obstacles are described by variations of the absurd subset. In other words, the quantum logical observer says, "The

[1]Department of Mathematics, N. A. Voznesensky Finance and Economics Institute, St. Petersburg 191023, Russia.

object is all that I could see!," while the binary logic observer asserts, "The object is all that I could do!." In this paper I show that the binary logic approach is rich enough. Namely, for any given ortholattice L the semigroup is explicitly built which may be the generating semigroup of a binary logic in such a way that the additive part of this binary logic is the ortholattice isomorphic to L. To do this, some tools to work with lattices are described.

In fact, dealing with lattices often needs some representation theorem. The most natural way to describe a lattice is to represent it as a lattice of sets. There are appropriate theorems for Boolean algebras (Stone theorem) and orthomodular lattices (Foulis, 1960). In this paper the means are suggested to "drag out" the underlying structure of finite ortholattices.

The first tool I propose is to represent a complete ortholattice L as the collection of closed subsets of a set V equipped with orthogonality relation $\perp$ [Section 1 and Zapatrin (1990*a*,*b*)].

The second tool is the representation of L as the annihilator ortholattice of a *-semigroup. In this paper I explicitly describe the construction of this generating *-semigroup by given ortholattice L. The construction is based on the two following ideas. The first one [which belongs to Foulis (1960)] is to consider the *-semigroup $S(L)$ of hemimorphisms of L admitting conjugation. The second one is to represent these hemimorphisms by binary relations on V.

1. POLARITIES AND TOPOLOGIES

The well-known source of complete lattices and ortholattices is the polarity construction (Birkhoff, 1967): let X, Y be two sets and O be a binary relation between X and Y:

$$O \subseteq X \times Y$$

The *right polar* (*left polar*) to a subset $A \subseteq X$ ($B \subseteq Y$) is the subset of Y (of X) defined as follows:

$$\begin{aligned} A^R &= \{y \in Y \mid \forall a \in A \;\; aOy\} \\ {}^L B &= \{x \in X \mid \forall b \in B \;\; xOb\} \end{aligned} \tag{1.1}$$

It is known that the collections R of right polars and L of left polars form the pair of anti-isomorphic complete ortholattices. The anti-isomorphism is realized by the pair of mappings (1.1) $(\cdot)^R\colon L \to R$ and ${}^L(\cdot)\colon R \to L$ (Birkhoff, 1967).

The special case when $X = Y = V$ and the relation, denote it $\perp$, is symmetric and irreflexive is very important. In this case $\perp$ will be called the *orthogonality* relation on V and for any subset $A \subseteq V$ its left and right polars do coincide:

$$^{L}A = A^{R} = A^{\perp}$$

Therefore the lattices R and L are isomorphic:

$$R = L = \Gamma_{\perp}(V) = \Gamma$$

and the anti-isomorphism between R and L becomes the orthocomplementation on Γ:

$$(\cdot)^{\perp}: \quad \Gamma \to \Gamma$$

So, let a set V be equipped with a symmetric ($x \perp y$ implies $y \perp x$), irreflexive (if $x \perp x$, then $x \perp y$ for any $y \in V$) relation called orthogonality. It will be assumed in the sequel that there is not more than one element $0 \in V$ such that $0 \perp y$ for any $y \in V$. The closure operation is defined for any $A \subseteq V$ as

$$\mathbf{Cl}\, A := A^{\perp\perp}$$

and possesses the following properties: for any $A, B \in V$:

C1. $A \subseteq \mathbf{Cl}\, A$.
C2. $\mathbf{Cl}\, \mathbf{Cl}\, A = \mathbf{Cl}\, A$.
C3. $A \subseteq B$ implies $\mathbf{Cl}\, A \subseteq \mathbf{Cl}\, B$.

So, Γ is the set of all *closed* ($A = \mathbf{Cl}\, A$) subsets.

A subset of V is called open if it is the set-theoretic complement in V of a closed subset. The interior **Int** A of a subset $A \subseteq V$ can be defined as the greatest open set contained in A. The interior operator **Int** possesses the following properties: for any $A, B \subseteq V$:

I1. $\mathbf{Int}\, A \subseteq A$.
I2. $\mathbf{Int}\, \mathbf{Int}\, A = \mathbf{Int}\, A$.
I3. $A \subseteq B$ implies $\mathbf{Int}\, A \subseteq \mathbf{Int}\, B$.

The collection of open sets can be considered as defining something like a topology on V. However, this is not the topology in a rigorous conventional sense, since the union of two closed sets (or intersection of two open ones) may not be closed (open).

Let $(L, ')$ be a complete ortholattice and V be its V-generating subset. V can be equipped with the orthogonality relation inherited from L:

$$x \perp y \quad \text{if and only if} \quad x \leq y'$$

Though, as shown in Zapatrin (1990*a,b*), the ortholattices L and Γ are isomorphic. Therefore in the sequel I shall consider ortholattices already represented as collections of closed sets.

2. ANNIHILATORS IN SEMIGROUPS

Let A be a semigroup with zero 0. Consider a binary relation O on A defined for any $a, b \in A$ as

$$aOb \quad \text{if and only if} \quad ab=0$$

The relation O is in general neither symmetric nor irreflexive, but it can be considered as a binary relation between two sets A^R and ${}^LA$ both isomorphic to A, and apply the polarity construction. Let $Q \subseteq A$. The right and left *annihilators* of Q are the sets

$$Q^R = \{x \in A \mid \forall q \in Q \;\; qx=0\}$$
$${}^LQ = \{y \in A \mid \forall q \in Q \;\; yq=0\}$$

Evidently any left (right) annihilator is a left (right) ideal of A. The closures of a subset $Q \subseteq A$ relative to the lattices L and R will be denoted by $\langle Q|$ and $|Q\rangle$, respectively:

$$\langle Q| := {}^L(Q^R) \in L, \qquad |Q\rangle := ({}^LQ)^R \in R$$

Now let A be a *-semigroup, i.e., there is an involution $(\cdot)^*: A \to A$ such that $t^{**}=t$ and $(ts)^* = s^*t^*$ for any $s, t \in A$. While the polarity construction yields the anti-isomorphism between L and R, the involution establishes the isomorphism of L and R, namely:

$$(\langle Q|)^* = |Q^*\rangle, \qquad (|Q\rangle)^* = \langle Q^*| \tag{2.1}$$

So, both isomorphic lattices L and R are endowed with the orthocomplementation:

$$\langle Q|^\perp = {}^L(|Q^*\rangle) \in L \qquad \text{and} \qquad |Q\rangle^\perp = (\langle Q^*|)^R \in R \tag{2.2}$$

Now the *annihilator ortholattice* $N(A)$ of a *-semigroup A can be defined as the ortholattice isomorphic to both R and L with the orthocomplementation (2.2). Note that while the ortholattices R and L are isomorphic, they do not coincide as in the case of symmetric orthogonality.

3. GENERATING *-SEMIGROUPS

Let $(L, ')$ be a complete ortholattice. A mapping $\phi: L \to L$ $(x \mapsto x\phi)$ is called monotone if it preserves partial order in L:

$$a \leq b \qquad \text{implies} \quad a\phi \leq b\phi$$

All the monotone mappings $\phi: L \to L$ form the semigroup with zero (under composition); denote it $M(L)$. A monotone mapping ϕ is called a *hemimorphism* of L if it preserves joins in L:

$$(x \vee y)\phi = x\phi \vee y\phi$$

All the hemimorphisms of L evidently form a subsemigroup of $M(L)$; denote it $E(L)$. Two mappings ϕ, $\psi \in E(L)$ are called *conjugated* if the following inequalities hold for any $x \in L$:

$$(x\phi)'\psi \leq x', \qquad x(\psi)'\phi \leq x'$$

It was proved by Foulis (1960) that if ϕ is conjugated with both χ and ψ, then $\chi = \psi$. So, if ϕ admits conjugation, the conjugated mapping is unique; denote it ϕ^+. Besides that, if a mapping ϕ admits conjugation, then both ϕ and ϕ^+ preserve joins in L; hence they are hemimorphisms: ϕ, $\phi^+ \in E(L)$. It follows immediately from (3.1) that $\phi^{++} = \phi$ and $(\phi\psi)^+ = \psi^+\phi^+$. Thus, all the hemimorphisms of L admitting conjugation form the semigroup with involution.

The *generating semigroup* $S(L)$ (or simply S, when no ambiguity occurs) of an ortholattice $(L, ')$ is the *-semigroup of all hemimorphisms of L admitting conjugation.

The *-semigroup $S(L)$ is called generating for L in virtue of the following theorem: *The annihilator ortholattice $N(S(L))$ is isomorphic to the ortholattice L.* The proof of the theorem consists of three stages:

1. Prove that any annihilator is the closure of an element of $S = S(L)$. Namely, for any $Q \subseteq S$ there exists such $p \in L$ that $\langle Q| = \langle \theta_p|$, where θ_p and its conjugate θ_p^+ are defined as follows:

$$x\theta_p = \begin{cases} p, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad x\theta_p^+ = \begin{cases} 0, & x \leq p' \\ I, & \text{otherwise} \end{cases}$$

where I is the greatest element of L.

2. Build the pair of mappings $F: L \to N(S)$ and $G: N(S) \to L$. For any $p \in L$, $Q \in N(S)$,

$$F(p) := \langle \theta_p| = \{\phi \mid I\phi \leq p\} \in N(S), \qquad G(Q) := \vee \{I\phi \mid \phi \in Q\} \in L$$

3. Prove that the pair F, G do realize the isomorphism between L and $N(S)$.

So, any complete ortholattice L is isomorphic to the annihilator ortholattice of the *-semigroup $S(L)$ of all its hemimorphisms admitting conjugation.

4. CLOSURES ON THE SEMIGROUP OF RELATIONS

Let V be a set and $\mathscr{B}$ be a semigroup of all binary relations on V with the product defined for any $T, S\in\mathscr{B}$ as

$$xTSy \quad \text{iff} \quad \exists z\in V \quad \text{such that} \quad xTz \quad \text{and} \quad zSy$$

and the transposition

$$xT^*y \qquad \text{if and only if} \quad yTx$$

The unit element 1 of the semigroup $\mathscr{B}$ is the equality relation. Moreover, since $\mathscr{B}$ is the collection of sets, the set-theoretic operations are defined for elements of $\mathscr{B}$:

$$T\cup S\in\mathscr{B}, \qquad \mathscr{T}\cap\mathscr{S}\in\mathscr{B}$$

$$\bar{T}=(V\times V)\backslash T\in\mathscr{B}$$

When the set V is endowed with an orthogonality $\perp$, two more operations can be defined on $\mathscr{B}$:

$$A^0:=\overline{AP}, \qquad {}^0A:=\overline{PA} \tag{4.1}$$

where $P=\bar{\perp}=\{(x,y)|\neg(x\perp y)\}$ is the relation of nonorthogonality. The partial order on V inherited from L in terms of the defined operations looks like

$$\leq\,=\overline{P\perp}={}^0\perp={}^{00}1, \qquad \geq\,=\overline{\perp P}=\perp^0=1^{00}$$

where 1 is the equality relation on $\mathscr{B}$. The operation $A\mapsto A^{00}$ on elements of $\mathscr{B}$ satisfies the conditions C1–C3 (Section 1); hence it is a closure on $\mathscr{B}$. The following fact is essential: a relation $A\in\mathscr{B}$ is ^{00}-closed if and only if for any $x\in V$ the set $\{y|xAy\}$ is the closed subset of $(V,\perp)$:

$$A=A^{00} \qquad \text{iff} \quad \forall x \quad \{y|xAy\}\in\Gamma_\perp(V)$$

In the sequel, the *monotone closure* operator $M\,\mathrm{Cl}\,A$ will be used:

$$M\,\mathrm{Cl}\,A=(\perp^0A)^{00} \tag{4.2}$$

It can be proved that $M\,\mathrm{Cl}$ is the closure of $\mathscr{B}$ and that M-closed relations are monotone on V:

$$A=M\,\mathrm{Cl}\,A \qquad \text{iff} \quad \forall x,y\in V \qquad x\leq y \quad \text{implies} \quad \{z|xAz\}\subseteq\{z|yAz\}$$

The collection $\mathscr{M}$ of all M-closed relations on $\mathscr{B}$ is the semigroup which is not a subsemigroup of $\mathscr{B}$, since, in general, the $\mathscr{B}$-product AB of two elements of $\mathscr{M}$ may not be an element of $\mathscr{M}$. The semigroup product of $\mathscr{M}$ is defined in the standard way:

$$A * B := M\,\mathrm{Cl}(AB) = (AB)^{00}$$

The defined product operation is associative since $\mathscr{M}$ is embedded to the semigroup $M(L)$, as it will be shown in the next section.

5. REPRESENTATION OF GENERATING SEMIGROUPS

As was mentioned in the Section 3, the elements of generating semigroup for the ortholattice L are hemimorphisms admitting conjugation. Let V be the $\vee$-generating subset of L. Any mapping $\phi: L \to L$ generates the binary relation R_ϕ on V defined as follows:

$$xR_\phi y \qquad \text{iff} \quad y \le x\phi \tag{5.1}$$

Conversely, given a binary relation R on V, the mapping ϕ_R: $V \to L$ can be defined as

$$x\phi_R = \vee\{y | xRy\}$$

which is extended to the mapping ϕ_R: $L \to L$:

$$a\phi_R = \vee\{x\phi_R | x \in V \quad \text{and} \quad x \le a\} \tag{5.2}$$

Evidently, the mapping (5.2) is always monotone. Now let $R \in \mathscr{B}$ be an arbitrary binary relation on V, and ϕ_R be the monotone mapping (5.2) associated with R. Consider the relation (5.1) associated with the mapping ϕ_R. In terms of operations on $\mathscr{B}$, this relation is the monotone closure of the initial relation R:

$$R_{\phi_R} = (\perp^0 R)^{00} = M\,\mathrm{Cl}\,R$$

Besides that, if a mapping ϕ is the hemimorphism, the mapping ϕ_R associated with the relation $R = R_\phi$ is equal to ϕ. So, the mapping $\phi \mapsto \phi_R$ establishes the injective mapping $E(L) \to M$, which is also the semigroup monomorphism. The conservation of the product is proved using the fact that $x\phi\psi = \vee\{y\psi | y \in x\psi\}$. The generating semigroup $S(\Gamma)$ is a subsemigroup of $E(\Gamma)$. Denote the image of $S(\Gamma)$ under the mapping (5.1) by $\mathscr{S}$. $\mathscr{S}$ is a subsemigroup of $\mathscr{M}$. In terms of relations on V, the conjugation is described as follows. If $\phi \in S(L)$ and $\psi = \phi^+$, then

$$R^0_\psi = R^{0*}_\psi \tag{5.3}$$

where $(\cdot)^0$ is the operation (4.1). [The proof uses the inequalities (3.1) in the form $x\in((x\phi)'\psi)'$ reformulated in terms of relations.] Consequently, the condition for a relation R to be an element of $\mathscr{S}$ is

$$R\in\mathscr{S} \quad \text{iff} \quad \mathscr{R}\in\mathscr{M} \quad \text{and} \quad \mathscr{R}'^{*}\in\mathscr{M}$$

Now define the operation $(\cdot)^{+}: \mathscr{M}\to\mathscr{M}$ as

$$R^{+}:=(\perp^{0}R^{0*})^{0}$$

The defined operation $T\mapsto T^{+}$ is not yet a conjugation in $\mathscr{M}$, since $T^{++}\subseteq T$ in general. However, the operator **Int** $T:=T^{++}$ possesses the properties I1–I3 (Section 1). Thus, due to (5.3) the elements of $\mathscr{S}$ are $^{++}$-open elements of $\mathscr{M}$, while the elements of $\mathscr{M}$ are, in turn, $\mathscr{M}$-closed (4.2) binary relations on V. This completes the description of $\mathscr{S}$: *The elements of the generating semigroup $\mathscr{S}$ are $^{++}$-open $\mathscr{M}$-closed relations on the $\vee$-generating set V.*

6. SUMMARY AND CONCLUDING REMARKS

Given a complete ortholattice $(L, ')$, it is associated with the collection of closed subsets of the set V with the orthogonality $\perp$.

This collection $\Gamma_{\perp}(V)$ is isomorphic to $(L, ')$ and always exists, as was treated in Zapatrin (1990*b*). The generating semigroup for the ortholattice $(L, ')$ is a *-semigroup whose annihilator ortholattice is isomorphic to $(L, ')$. As was shown in this paper, such *-semigroups do always exist and can be isomorphically represented as the semigroup $\mathscr{S}$ whose elements are binary relations on the set V. The main result of the paper is the obtained representation of the semigroup $\mathscr{S}$. The elements of $\mathscr{S}$ are obtained twice applying the closure construction to the set $\mathscr{B}$ of all binary relation on V. In turn, this semigroup $\mathscr{S}$ can be considered as the generating semigroup of a binary logic (Zapatrin, 1989). In this case the initial lattice L will be (isomorphic to) the additive logic of the generating semigroup $(\mathscr{S}, \{'\})$.

I see the following areas of possible applications of the tools proposed:

1. This description is more compact and brings more information about the structure of the ortholattice than, for example, traditional Hasse diagrams.
2. A measure defined on an ortholattice can be represented as a function on the set or semigroup generating the ortholattice.
3. The representation of ortholattices by collections of sets could provide "fuzzying" of the reasonings without introducing new essences like "fuzzy lattices."
4. It may be a step toward describing varying topology, which can be very useful in quantizing gravity (Zapatrin, 1991).

ACKNOWLEDGMENTS

The author is greatly indebted to C. Garola, G. Nisticò, and G. Cattaneo for discussions and remarks, and to T. Chuklina and Dima Karateev for the initial preparation of the manuscript.

REFERENCES

Birkhoff, G. (1967). *Lattice Theory*, American Mathematical Society, Providence, Rhode Island.

Foulis, D. (1960). Baer *-semigroups, *Proceedings of the American Mathematical Society*, **11**, 648.

Girard, J.-Y. (1987). Linear logic, *Theoretical Computer Science*, **50**, 1.

Zapatrin, R. R. (1989). *International Journal of Theoretical Physics*, **28**, 1323.

Zapatrin, R. R. (1990*a*). Graphs, polarities and finite ortholattices, in *Proceedings of the Second Winter School on Measure Theory*, pp. 213–218, A. Dvurecenskij and S. Pulmannova, eds., SAV, Bratislava.

Zapatrin, R. R. (1990*b*). Graph representation of finite ortholattices, *Discrete Mathematics*, submitted.

Zapatrin, R. R. (1991). Pre-Regge calculus: Topology via logic, *Classical and Quantum Gravity*, submitted.